ISRAEL JOURNAL OF MATHEMATICS 94 (1996), 367–379

A CHARACTERISTIC SUBGROUP OF Σ_4-FREE GROUPS

BY

BERND STELLMACHER

Christian-Albrechts-Universität zu Kiel
Ludewig-Meyn-Str. 4, D-24098 Kiel, Germany

ABSTRACT

Let S be a finite non-trivial 2-group. It is shown that there exists a non-trivial characteristic subgroup $W(S)$ in S satisfying: $W(S)$ is normal in H for every finite Σ_4-free group H with $S \in \mathrm{Syl}_2(H)$ and $C_H(O_2(H)) \leq O_2(H)$.

Let S be a non-trivial finite 2-group. In [Gl$_1$] (see also [Gl$_2$]) Glauberman raised the question whether it is possible to find a non-trivial characteristic subgroup $W(S)$ of S such that $W(S)$ is normal in H for every finite group H satisfying:

(I) H is Σ_4-free,

(II) $S \in \mathrm{Syl}_2(H)$ and $C_H(O_2(H)) \leq O_2(H)$.

An affirmative answer to this question would provide an analogue to Glauberman's ZJ-Theorem for the prime 2. It also would improve Glauberman's Triple-Factorization Theorem proved in [Gl$_2$].

In this note we will give an answer to that question under the following additional hypothesis (which was also used in [Gl$_2$]):

(III) Every non-abelian simple section of H is isomorphic to $\mathrm{Sz}(2^m)$ or $\mathrm{PSL}_2(3^m)$ for some odd m.

In section 3 we will define a characteristic subgroup $W(S)$ of S with $\Omega_1(Z(S)) \leq W(S) \leq Z(J(S))$ for which the following theorem holds.

THEOREM: *Suppose that H is a finite group satisfying (I), (II) and (III). Then $W(S)$ is normal in H.*

Received October 17, 1993 and in revised form April 7, 1994

As with the Triple-Factorization Theorem, one can use the above Theorem and Goldschmidt's result about groups containing a strongly closed abelian 2-subgroup [Gol] to show that the hypotheses (I) and (II) already imply hypothesis (III) (see section 7 in [Gl_2]).

The proof of the Theorem uses the same approach as in [St_1] via embeddings and amalgamated products, but in contrast to [St_1] it does not use the amalgam method.

The properties of the groups $\mathrm{Sz}(2^m)$ and their GF(2)-modules and of the groups $L_2(3^m)$ used in the proof can be found in [Su], [Ma], [Gl_2] and [Hu], respectively. If the reader restricts himself to solvable groups H the proof becomes, apart from textbook material, self-contained.

1. Embeddings

Let p be a prime and S a finite p-group. An **embedding** of S is a pair (τ, H) where H is a group and τ a monomorphism from S into H. We are interested in the following class of embeddings of S and certain subclasses of it.

Let $\mathcal{C}$ be the class of all embeddings (τ, H) of S such that

(i) H is finite and $S\tau \in \mathrm{Syl}_p(H)$, and

(ii) $C_H(O_p(H)) \leq O_p(H)$.

Let $\mathcal{U}$ be a non-empty subclass of $\mathcal{C}$. Then $\mathcal{U}$ is **characteristically closed**, if

(*) $(\alpha\tau, H) \in \mathcal{U}$ for every $(\tau, H) \in \mathcal{U}$ and $\alpha \in \mathrm{Aut}(S)$.

$O_S(\mathcal{U})$ denotes the largest subgroup X of S such that

(**) $X\tau$ is normal in H for every $(\tau, H) \in \mathcal{U}$.

1.1: *Let $\mathcal{U}$ be a characteristically closed subclass of $\mathcal{C}$ and $\alpha \in \mathrm{Aut}(S)$. Suppose that the subgroup $X \leq S$ satisfies (**). Then $X\alpha$ satisfies (**). In particular, $O_S(\mathcal{U})$ is a characteristic subgroup of S.*

Proof: Let $(\tau, H) \in \mathcal{U}$. Then $(\alpha\tau, H) \in \mathcal{U}$ and thus $X(\alpha\tau) = (X\alpha)\tau$ is normal in H. ∎

Let $(\tau_i, H_i) \in \mathcal{U}$ for $i = 1, 2$. We define (τ_1, H_1) and (τ_2, H_2) to be **equivalent**, if there exists an isomorphism ϕ from H_1 to H_2 such that $\tau_1\phi = \tau_2$. This defines an equivalence relation on $\mathcal{U}$. Let $[\mathcal{U}]$ be the class of equivalence classes of $\mathcal{U}$ with respect to this equivalence relation.

1.2: *Let $\mathcal{U}$ be a subclass of $\mathcal{C}$. Then $[\mathcal{U}]$ is a finite set.*

Proof: By (i) and (ii) in the definition of $\mathcal{C}$ the order of H is bounded by a function of $|S|$. Hence, if S is fixed, there exist only finitely many non-isomorphic groups H with (i) and (ii); and for every such group H there exist only finitely many monomorphisms from S into H. ∎

Let $\{(\tau_1, H_1), \dots, (\tau_r, H_r)\}$ be a set of representatives of the classes in $[\mathcal{U}]$. Then the amalgamated product of the groups $H_1, ..., H_r$ over S (for the definition see [Se]) is unique up to isomorphism and does not depend on the choice of the representatives. We denote this group by $G(\mathcal{U})$.

As usual we identify the groups $S, H_1, \dots, H_r$ with their images in $G(\mathcal{U})$.

1.3: *Let $\mathcal{U}$ be a non-empty subclass of $\mathcal{C}$ and X a subgroup of S. Then the following two statements are equivalent:*

(a) *X satisfies (**).*

(b) *X is normal in $G(\mathcal{U})$.*

In particular, $O_S(\mathcal{U})$ is the largest subgroup of S which is normal in $G(\mathcal{U})$.

Proof: This follows directly from the definition of $G(\mathcal{U})$ and the identifications we have made. ∎

1.4: *Let $\mathcal{U}$ be a characteristically closed subclass of $\mathcal{C}$, A a characteristic subgroup of S in $O_S(\mathcal{U})$ and $W = \langle A^{G(\mathcal{U})}\rangle$. Then W is a characteristic subgroup of S.*

Proof: Let $\alpha \in \mathrm{Aut}(S)$. By 1.3, W satisfies (**). Hence, by 1.1, $W\alpha$ satisfies (**) and, again by 1.3, $W\alpha$ is normal in $G(\mathcal{U})$. Since $A\alpha = A \leq W\alpha$ we get $W \leq W\alpha$ and then $W = W\alpha$. ∎

Let $\mathcal{C}_M$ be the class of all embeddings $(\tau, H) \in \mathcal{C}$ such that:

(M) For every normal subgroup N of H, either $S\tau \cap N \leq O_p(H)$ or $O^p(H) \leq N$.

1.5: *Let $(\tau, H) \in \mathcal{C}$. Then there exist subgroups $H_1, \dots, H_n$ of H containing $S\tau$ such that $H = \langle H_1, \dots, H_n\rangle$ and $(\tau, H_i) \in \mathcal{C}_M$ for $i = 1, \dots, n$.*

Proof: We proceed by induction on $|H|$, and we identify S with its image in H. Let N be a normal subgroup of H. Then $H = N_H(S \cap N)N$. If both $N_H(S \cap N)$ and SN are proper subgroups, then by induction the assertion holds for these subgroups and thus for H. Hence, we may assume that $H = N_H(S \cap N)$ or $H = SN$ for every normal subgroup N of H. But then H satisfies (M). ∎

A group H is X-**free**, if $A/B \not\cong X$ for every subgroup A of H and every normal subgroup B of A. It is easy to see that a finite group H satisfying (II) is Σ_4-free, if and only if $H/O_2(H)$ is Σ_3-free.

2. Modules

In this section let p be a prime, S a finite p-group and V a finite $\mathrm{GF}(p)S$-module. We define

$$\mathcal{E}(S,V) = \{A \leq S \mid [V,A] \neq 1,\ A/C_A(V) \text{ elementary abelian}\};$$
$$a(S,V) = \infty, \quad \text{if } \mathcal{E}(S,V) = \emptyset,$$
$$a(S,V) = \min\{\log_{|A/C_A(V)|}(|V/C_V(A)|) \mid A \in \mathcal{E}(S,V)\}, \text{ otherwise.}$$

2.1: *Let $A \in \mathcal{E}(S,V)$ and U be a subspace of V. Suppose that $|A/C_A(V)|^e = |V/C_V(A)|$. Then*

$$\textit{either } |U/C_U(A)| \leq |A/C_A(U)|^e \textit{ or } |V/C_V(C_A(U))| < |C_A(U)/C_A(V)|^e.$$

Proof: Set $A_0 = C_A(U)$. Assume that $|U/C_U(A)| > |A/A_0|^e$. Then

$$\begin{aligned} |V/C_V(A_0)| &\leq |V/C_V(A)||C_V(A)U/C_V(A)|^{-1} \\ &< |A/C_A(V)|^e|A/A_0|^{-e} = |A_0/C_A(V)|^e. \end{aligned}$$ ∎

2.2: *Let $A \in \mathcal{E}(S,V)$ such that $|A/C_V(A)|^{a(S,V)} = |V/C_V(A)|$. Then for every A-submodule U of V, either $a(A,U) \leq a(S,V)$ or $[U,A] = 1$.*

Proof: Let U be an A-submodule of V such that $[U,A] \neq 1$, and let $A_0 = C_A(U)$. Then 2.1 implies that $a(A,U) \leq a(S,V)$, or

$$|V/C_V(A_0)| < |A_0/C_A(V)|^{a(S,V)} = |A_0/C_{A_0}(V)|^{a(S,V)}.$$

The second possibility contradicts the definition of $a(S,V)$. ∎

In the following let H be a finite group, $S \in \mathrm{Syl}_p(H)$ and V a finite $\mathrm{GF}(p)H$-module. Moreover, we assume that $V = \langle Z_0^H \rangle$ for some S-submodule Z_0 of V. Set $a = a(S,Z_0)$ and $\overline{H} = H/C_H(V)$.

2.3 ([St_2]): *Suppose that there exists* $A \in \mathcal{E}(S,V)$ *such that* $[Z_0, A] = 1$ *and* $A \le O_p(H)$. *Then* $|\overline{A}|^a \le |V/C_V(A)|$.

Proof: We define subgroups $A = A_0 \ge A_1 \ge \cdots \ge A_t = C_A(V)$ and conjugates $W_i = Z_0^{x_i}$, $x_i \in H$, such that $A_i = C_{A_{i-1}}(W_i)$ for $i = 1, \ldots, t$. Then

$$|A_{i-1}/A_i|^a \le |W_i/C_{W_i}(A_{i-1})|$$

and

$$|\overline{A_0}|^a \le \prod_{i=1}^{t} |W_i/C_{W_i}(A_{i-1})|.$$

On the other hand

$$|W_i/C_{W_i}(A_{i-1})| = |W_iC_V(A_{i-1})/C_V(A_{i-1})| \le |C_V(A_i)/C_V(A_{i-1})|,$$

and thus

$$|C_V(A_t)/C_V(A_{t-1})| \cdots |C_V(A_1)/C_V(A_0)| = |V/C_V(A_0)| \ge \prod_{i=1}^{t} |W_i/C_{W_i}(A_{i-1})| \ge |\overline{A_0}|^a. \quad \blacksquare$$

2.4: *Let* $p = 2$ *and* M *be a maximal subgroup of* H *containing* S. *Suppose that*

(i) *Every non-abelian simple section of* H *is isomorphic to* $\mathrm{Sz}(2^m)$ *or* $L_2(3^m)$ *for some odd* m,

(ii) H *is* Σ_3*-free, and*

(iii) A *is an elementary abelian subgroup of* S, *and* $A \not\le \bigcap_{h \in H} M^h$.

Then there exists a subgroup L *of* H *such that*

(a) $A \le S \cap L \in \mathrm{Syl}_2(L)$ *and* $L \not\le M$,

(b) $L = \langle A, A^x \rangle$ *for every* $x \in L \setminus M$, *and*

(c) $L/O_2(L) \cong D_{2r^s}$, r *an odd prime, or* $L/O_{2,2'}(L) \cong \mathrm{Sz}(2^m)$, $m > 1$.

Proof: We proceed by induction on $|H|$. Then we may assume that $O_2(H) = 1$. Let $D = \bigcap_{h \in H} M^h$ and $\tilde{H} = H/D$, and let $|U|$ be minimal with $S \le U \le H$ and $H = UD$. Then $M \cap U$ is a maximal subgroup of U and

$$\bigcap_{u \in U} (M^u \cap U) \le \bigcap_{u \in U} M^u = D.$$

Hence, U satisfies the hypothesis with respect to $M \cap U$, and by induction we may assume:

$$(*) \qquad \text{If } S \le U \text{ and } H = UD, \text{ then } U = H.$$

In particular, the Frattini argument and (*) imply that $O_2(\tilde{H}) = 1$ and D has odd order.

Assume that $D \neq 1$. Then by induction there exists $\tilde{L} \leq \tilde{H}$ satisfying (a)–(c) with respect to $\tilde{S}$, $\tilde{A}$ and $\tilde{M}$. Hence, L satisfies the hypothesis with respect to $L \cap M$, and again by induction we may assume that $L = H$. Now (*) and $O_2(\tilde{H}) = 1$ show that L satisfies (a)–(c).

Assume now that $D = 1$. Let N be a minimal normal subgroup of H. Note that $H = NM$. Suppose that N is solvable. Then N has odd order, and

$$[N, A] = \langle [C_N(A_0), A] \,|\, |A/A_0| = 2 \rangle.$$

If $[N, A] \leq M$, then $\langle A^H \rangle \leq M$, a contradiction. Thus, there exists a subgroup $A_0 \leq A$ such that $|A/A_0| = 2$ and $[C_N(A_0), A] \not\leq M$. A subgroup L satisfying (a)–(c) is now easy to find in $[C_N(A_0), A]A$.

Suppose that N is not solvable. Then $N = E_1 \times \cdots \times E_s$, $E_i \cong \mathrm{Sz}(2^m)$ or $L_2(3^m)$. In both cases $N_H(E_i)/E_iC_H(E_i)$ has odd order, either by (ii) or since $\mathrm{Aut}(\mathrm{Sz}(2^m))/\mathrm{Sz}(2^m)$ has odd order. Hence $N_A(E_i) \leq E_iC_H(E_i)$, and as above $[N, A] \not\leq M$. Hence, we may assume that $[E_1, A] \not\leq M$. Let A_1 be a subgroup of A such that

(*) $A_1 \times N_A(E_1) = A$, if $C_A(E_1) \neq N_A(E_1)$, and

(**) $C_A(E_1) \leq A_1$ and $|A/A_1| = 2$, if $C_A(E_1) = N_A(E_1)$.

Define $E = \{\prod_{a \in A_1} e^a | e \in E_1\}$. Then $E \cong E_1$, E is a normal subgroup of $C_N(A_1)$, and $[E, A] \neq 1$. Moreover, $[E, E_1 \cap S] = E_1 \not\leq M$ shows that $E \not\leq M$.

Suppose that $A \not\leq N_H(E)$. Then (**) holds, and $[E, a] = E \times E^a$ for $a \in A \setminus A_1$. Hence, there exists an element $x \in E$ of odd prime order such that $x^{-1}x^a \notin M$. Now $L = A\langle x^{-1}x^a \rangle$ satisfies (a)–(c).

Suppose that $A \leq N_H(E)$. Note that $EA = EZ(EA)$. If $E \cong \mathrm{Sz}(2^m)$, the assertion is easy to check in EA. Assume that $E \cong L_2(3^m)$, m odd. By [Hu, II.8.27] we may assume that $(EA \cap M)Z(EA)/Z(EA) \cong D_{3^m+1}$ and $AZ(EA) \in \mathrm{Syl}_2(EA)$. But then $N_E(A) \not\leq MZ(EA)$, and the assertion can be verified in $(M \cap EA)^y$ for $y \in N_E(A) \setminus MZ(EA)$. ∎

2.5: *Let* $p = 2$ *and* A, M, H *and* L *be as in 2.4, and let* $W = \langle Z_0^L \rangle$ *and* $\tilde{L} = L/C_L(W)$. *Suppose that*

(i) $N_H(Z_0) \leq M$ *and* $[Z_0, A] = 1$, *and*

(ii) $[V, A, A] = 1$.

Then the following hold:

(a) $W = C_W(A)C_W(A^x)$ *for* $x \in L\backslash M$.

(b) $C_W(A) = [W,A]C_W(L) = [W,a]C_W(L) = C_W(a)$ *for* $a \in A\backslash O_2(L)$.

(c) *If* $|A/A \cap O_2(L)| = 2$, *then* $\tilde{L}/O_2(\tilde{L}) \cong D_{2r^s}$ *and every chief factor of* $\tilde{L}$ *in* $W/C_W(L)$ *is non-central.*

(d) *If* $|A/A\cap O_2(L)| \geq 4$, *then* $\tilde{L}/O_2(\tilde{L}) \cong \mathrm{Sz}(2^m), m > 1$, *and every chief factor of* $\tilde{L}$ *in* $W/C_W(L)$ *is a natural* $\mathrm{Sz}(2^m)$*-module. Moreover, if* $O_2(\tilde{L}) = 1$, *then* $W/C_W(L)$ *is the direct product of natural* $\mathrm{Sz}(2^m)$*-modules.*

Proof: Let $\overline{W} = W/C_W(L)$ and $a \in A\backslash O_2(L)$. Then there exists $y \in L$ such that $a^y \notin M$ and $L = \langle A, a^y\rangle$. Hence $\overline{W} = [\overline{W},A][\overline{W},a^y]\overline{Z}_0$. By our hypotheses $[W,A]Z_0 \leq C_W(A)$ and $[W,A]Z_0 \cap [W,a^y] \leq C_W(L)$. Thus

$$[\overline{W},A]\overline{Z}_0 \cap [\overline{W},a^y] = 1 \quad \text{and} \quad |[\overline{W},A]\overline{Z}_0|^2 \leq |\overline{W}|.$$

It follows that

$$|[\overline{W},a^y]| \leq |[\overline{W},A]| \leq |[\overline{W},A]\overline{Z}_0| \leq |\overline{W}/[\overline{W},A]\overline{Z}_0| = |[\overline{W},a^y]|.$$

This gives $[\overline{W},A] = [\overline{W},A]\overline{Z}_0$ and $\overline{W} = [\overline{W},A] \times [\overline{W},a^y]$. Now (a)–(c) follow.

We now assume that $|A/A \cap O_2(L)| \geq 4$. Then $L/O_{2,2'}(L) \cong \mathrm{Sz}(2^m), m > 1$. Suppose that $[O_{2,2'}(L),A] \not\leq O_2(L)$. Let $A_0 \leq A$ be maximal with $a \in A_0$ and $A_0 \cap O_2(L) = 1$. By (b), $[\overline{W},A_0] = [\overline{W},a]$. Now $|A_0| \geq 4$ shows that $O_{2,2'}(L)$ operates trivially on every chief factor of L in $\overline{W}$, and $O^2(O_{2,2'}(L)) \leq C_L(W)$. Since the Schur multiplier of $\mathrm{Sz}(2^m)$ is a 2-group this gives $\tilde{L}/O_2(\tilde{L}) \cong \mathrm{Sz}(2^m)$.

Let $\tilde{f}$ be an element of order 5 in $\tilde{L}$. Then $f \notin M$ since $N_L(S \cap L)O_{2,2'}(L) = L \cap M$. We may assume that f is inverted by a. It follows that $\overline{W} = [\overline{W},a] \times [\overline{W},a^f]$, and f operates fixed-point-freely on $\overline{W}$. Hence, [Ma] gives the remaining assertion of (d). ∎

2.6: *Suppose that* $p = 2$, $O_2(\overline{H}) = 1$ *and* H *satisfies* (i) *and* (ii) *of* 2.4. *Then* $2 \leq a(S,V)$.

Proof: This is Theorem A in the appendix of [Gl_2]. ∎

Let $H \cong \mathrm{Sz}(2^m)$ and V be a natural $\mathrm{Sz}(2^m)$-module. In the next lemma we use the following properties of V. There exists a series $1 = V_0 \leq \cdots \leq V_4 = V$ of $N_H(S)$-submodules of V such that

(i) $[V_i, S] = V_{i-1}$, $|V_i/V_{i-1}| = 2^m$, and V_i/V_{i-1} is an irreducible $N_H(S)$-module for $1 \leq i \leq 4$,

(ii) $V_1 = C_V(s) = [V_2, s]$ for every $s \in S \setminus Z(S)$, and $V_2 = C_V(z) = [V, z]$ for every $1 \neq z \in Z(S)$.

2.7: *Suppose that $H/Z(H) \cong \mathrm{Sz}(2^m)$, $m > 1$, and $Z(H) \leq H'$. Let W be a normal subgroup of $N_H(S)$ in S. Then either $Z(H) \leq W$ or $W \leq Z(H)$.*

Proof: By 17.4, 25.1 and 25.3 of [Hu], $Z(H) \leq S'$. Let W be a counterexample. Then we may assume that $|Z(H)/Z(H) \cap W| = 2$. Hence, the action of $N_H(S)$ on S shows that S/W is extraspecial. But $|S/WZ(H)| = 2^m$, m odd, a contradiction. ∎

2.8: *Let $H/O_2(H) \cong \mathrm{Sz}(2^m)$, $m > 1$, and let W be an elementary abelian normal subgroup of $N_H(S)$ in S. Suppose that*

(a) *$C_H(O_2(H)) \leq O_2(H)$ and $W \not\leq O_2(H)$, and*

(b) *$O_2(H)/Z(H)$ is the direct product of natural $\mathrm{Sz}(2^m)$-modules.*

Then there exists a subgroup $Q \leq S$ such that $\Phi(Q) \leq C_H(W)$ and $|W/C_W(Q)| \leq |Q/C_Q(W)| \neq 1$.

Proof: Let $X = O_2(H)$ and $\overline{X} = X/Z(H)$, and set $X_0 = [X, S]$ and $X_1 = [X_0, W]$. Note that $\overline{X} = (\overline{X \cap W})(\overline{X \cap W})^t$ for $t \in H \setminus N_H(S)$ and $|S/WX| = 2^m$. Note further that $|\overline{X}| = q^4$, $|\overline{X_0}| = q^3$ and $|C_{\overline{X}}(S)| = q$, where $q = 2^{mn}$, n the number of H-chief factors in $\overline{X}$.

Since $[\overline{X_0}, S, W] = [S, W, \overline{X_0}] = 1$ the 3-subgroup Lemma implies $[X_1, S] \leq Z(H)$ and $|\overline{X_1}| = q$. In addition, $\overline{X_0} = \{v \in \overline{X} | [v, w] \in \overline{X_1}\}$ for every $w \in W \setminus X$. Let $x \in X \setminus X_0$. Then $[X_1, x, S] = [S, X_1, x] = 1$ and thus also $[x, S, X_1] = 1$. Since $\langle [\overline{x}, S](\overline{W \cap X}) | x \in X \setminus X_0 \rangle = \overline{X_0}$, we conclude that $[X_1, X_0] = 1$.

Let $s \in S \setminus WX$. Then $s^2 = wv$ where $v \in (W \cap X)^t Z(H)$ and $w \in W$. It follows that $(wv)^2 \in [w, v]Z(H) = s^4 Z(H)$; in particular, $[w, v] \in C_{W \cap X}(s) \leq X_1 Z(H)$. But this implies that $v \in X_0 Z(H)$ and $s^2 \in WX_0 Z(H)$. Then there exists $v' \in X$ such that $[s, v'](W \cap X)Z(H) = v(W \cap X)Z(H)$. Hence $(sv')^2 = s^2 v'^2 [s, v'] \in w(W \cap X)Z(H)$. Thus, there exists $s \in S \setminus WX$ such that $s^2 \in WZ(H)$.

Note that $[s, W] \leq C_W(s) \cap X \leq X_1$ because W is elementary abelian and $s^2 \in WZ(H)$. Hence $[X_0, s, W] = [s, W, X_0] = 1$ and thus also $[W, X_0, s] = [X_1, s] = 1$. Let $Q = Z(H)X_0 W\langle s^g | g \in N_H(S)\rangle$. Then $[X_1, Q] = 1$ and $[W, Q] \leq$

X_1. It follows that $\Phi(Q) \le C_H(W)$. Moreover $C_W(Q) = X_1(Z(H) \cap W)$ and $C_Q(W) = WZ(H)$. This gives

$$|W/C_W(Q)| = 2^m q \le |Q/C_Q(W)|$$

since $QX = S$. ∎

2.9: *Suppose that* $p = 2$ *and* H *satisfies* (i) *and* (ii) *of* 2.4. *Then* $a(S, V) \le 1$ *implies* $a(S, Z_0) \le 1$.

Proof: Assume that $a(S, V) \le 1$ but $a(S, Z_0) > 1$, and let $|V/Z_0||H|$ be minimal with that property. Then $Z_0 \ne V$. We choose the following notation:

$\mathcal{E}^*(S, V)$ is the set of all $A \in \mathcal{E}(S, V)$ such that $|A/C_A(V)|^{a(S,V)} = |V/C_V(A)|$,

$\mathcal{E}_0^*(S, V)$ is the set of all $A \in \mathcal{E}^*(S, V)$ such that $|A/C_A(V)|$ is minimal,

$J(S, V) = \langle A | A \in \mathcal{E}_0^*(S, V)\rangle$, $M_0 = N_H(J(S, V))$, $Z_0^* = \langle Z_0^{M_0}\rangle$ and $M = N_H(Z_0^*)$.

From 2.2 we get that $[Z_0, J(S, V)] = 1$ and thus also $[Z_0^*, J(S, V)] = 1$. In particular $V \ne Z_0^*$ and $M \ne H$. Now the minimality of $|V/Z_0||H|$ gives either $Z_0 = Z_0^*$ or $a(S, Z_0^*) \le 1$. In the second case $|Z_0^*/Z_0||M_0| < |V/Z_0||H|$, and again the minimality of $|V/Z_0||H|$ implies that $a(S, Z_0) \le 1$, a contradiction. Thus, we have $Z_0 = Z_0^*$.

Let $M \le H_0 \le H$ such that M is a maximal subgroup of H_0, and let $Z_1 = \langle Z_0^{H_0}\rangle$. Then $Z_1 \ne Z_0$, and as above the minimality of $|V/Z_0||H|$ first gives $a(S, Z_1) \le 1$ and then $H_0 = H$.

Let $D = \bigcap_{h\in H} M^h$. Assume that $J(S, V) \le D$. Then by the Frattini argument $H = M_0 D \le M$, a contradiction. Hence, there exists $A \in \mathcal{E}_0^*(S, V)$ such that $A \not\le D$. Since $|A/C_A(V)|$ is minimal the Thompson-Replacement Theorem [Go] gives $[V, A, A] = 1$. We have shown that Z_0, A, M and H satisfy the hypotheses of 2.4 and 2.5. Moreover $M = N_H(Z_0)$.

Let L be as in 2.4. Set $U = \langle Z_0^L\rangle$, $\overline{U} = U/C_U(L)$, $A_0 = A \cap O_2(L)$, $Q = \langle A_0^L\rangle$ and $\overline{L} = L/C_L(U)$. Note that by 2.2, $a(A, U) \le a(A, V) \le 1$. Note further that $M \cap L = N_L(Z_0)$ and $[Z_0, A] = 1$; in particular $[N, U] = 1$ for every normal subgroup N of L in M with $[N, A] = N$. Thus, if $L/C_L(U)$ is solvable, then 2.5(c) and the Frattini argument give $M \cap L = C_L(U)(S \cap L)$. If $L/C_L(U)$ is not solvable, then 2.5(b), 2.5(d) and the structure of $\mathrm{Sz}(2^m)$ yield $M \cap L = C_L(U)N_L(S \cap L)$. Hence, in both cases $M \cap L = C_L(U)N_L(S \cap L) = N_L(Z_0)$.

By 2.5(a) we also have $U = C_U(A)C_U(A^x)$ for $x \leq L \setminus M$. For $y \in L$ we get that $[Z_0^y, A_0] \leq C_U(A)$ since $[U, A, A] = 1$, and $[Z_0^y, A_0] \leq Z_0^y \leq C_U(A^y)$. Hence

$$[Z_0^x, A_0] \leq C_U(A) \cap C_U(A^x) = C_U(L) \quad \text{for } x \in L \setminus M.$$

In particular, Q operates quadratically on U and $[U, Q] \leq C_U(L)$.

Let $B_0 = A_0[Q, A]$. Then $C_U(A) \leq C_U(B_0)$ since $[C_U(A), Q] \leq C_U(L)$, and $\overline{Q} = \overline{B}_0 \times \overline{B}_0^x$. Moreover, 2.6 (applied to L and a non-central L-chief factor of U) and $a(A, U) \leq 1$ imply that $\overline{A}_0 \neq 1$ and thus $\overline{B}_0 \neq 1$.

Assume that $|A/A_0| \geq 4$. Then by 2.5, $\overline{L}/O_2(\overline{L}) \cong \mathrm{Sz}(2^m)$ and every chief factor of $\overline{L}$ in $\overline{Q}$ and $\overline{U}$ is a natural $\mathrm{Sz}(2^m)$-module. Hence $|\overline{B_0}| = 2^{2km}$ and

$$|U/C_U(A)| = 2^{2sm} \leq |\overline{A}| \leq |A/A_0||\overline{B_0}| \leq 2^{2km+m}$$

since $a(A, U) \leq 1$. It follows that $k \geq s$ and $|\overline{B_0}| \geq |U/C_U(\overline{B_0})|$, and by 2.3 applied to B_0, U and L (in place of A, V and H)

$$|\overline{B}_0|^{a(S,Z_0)} \leq |U/C_U(B_0)| \leq |\overline{B}_0|.$$

Hence $a(S, Z_0) \leq 1$, a contradiction.

Assume now that $|A/A_0| = 2$. Then $\overline{L}/O_2(\overline{L}) \cong D_{2r}$, r an odd prime, since $\overline{M \cap L} = N_{\overline{L}}(\overline{S \cap L})$. Now $|\overline{Q}| = 2^{2k}$ and $|\overline{U}| = 2^{2s}$ where $r | 2^{2k} - 1$ and $r | 2^{2s} - 1$. On the other hand, as above

$$|U/C_U(A)| = 2^s \leq 2|\overline{B_0}| = 2^{k+1}$$

since $a(S, V) \leq 1$, and thus $s \leq k + 1$. If $s \leq k$, we get as above with 2.3 $a(S, Z_0) \leq 1$. Hence $s = k + 1$ and $2^{2k} \equiv 2^{2k+2} \equiv 1 \bmod r$. This gives $r = 3$, and L is not Σ_4-free, a contradiction. ∎

3. A characteristic subgroup

In this section $p = 2$ and $\mathcal{F}$ is the class of all embeddings (τ, H) of $\mathcal{C}$ such that H is Σ_4-free. Let $\mathcal{F}^*$ be the class of those (τ, H) in $\mathcal{F}$ satisfying:

(*) Every non-abelian simple section of H is isomorphic to $\mathrm{Sz}(2^m)$ or $L_2(3^m)$ for some odd m.

Note that condition (ii) in the definition of $\mathcal{C}$ implies that $H/O_2(H)$ is Σ_3-free for every $(\tau, H) \in \mathcal{F}$.

Throughout this section we identify S with its image in the corresponding groups H. We now define the following subclasses of $\mathcal{F}^*$:

$$\mathcal{F}_1\text{: all } (\tau, H) \in \mathcal{F}^* \cap \mathcal{C}_M \text{ with } J(S) \text{ normal in } H,$$

$$\mathcal{F}_2\text{: all } (\tau, H) \in \mathcal{F}^* \cap \mathcal{C}_M \text{ with } J(S) \text{ not normal in } H.$$

Let $G_1 = G(\mathcal{F}_1)$, $W_0 = \Omega_1(Z(S))$, and $W = \langle W_0^{G_1} \rangle$. Note that $W \leq Z(J(S))$ and that by 1.4 W is a characteristic subgroup of S.

For a sequence $(\tau_1, H_1), \ldots, (\tau_k, H_k)$ of elements of $\mathcal{F}_1$ we define recursively:

$$W_i = \langle W_{i-1}^{H_i} \rangle \quad \text{for } i = 1, \ldots, k.$$

Since $Z(J(S))$ is normal in G_1 there exists a sequence $(\tau_0, H_0), \ldots, (\tau_t, H_t)$ of elements of $\mathcal{F}_1$ such that

(i) $H_0 = S$, and

(ii) $W_i \neq W_{i-1}$ for $1 \leq i \leq t$ and $W_t = W$.

3.1: *$a(S, W_i) > 1$ for $0 \leq i \leq t$.*

Proof: Note that $a(S, W_0) = \infty$. Thus, 3.1 follows from 2.9. ∎

3.2: *Let $(\tau, H) \in \mathcal{F}_2$. Then either $[W_i, O^2(H)] = 1$ or $W_i \not\leq O_2(H)$.*

Proof: Suppose that $[W_i, O^2(H)] \neq 1$ and $W_i \leq O_2(H)$. Set $Y = \langle W_i^H \rangle$. Assume that Y is abelian. Then 2.9 and 3.1 give $[Y, J(S)] = 1$. Hence $[Y, O^2(H)] = 1$ since $J(S) \not\leq O_2(H)$, and thus also $[W_i, O^2(H)] = 1$.

Assume now that Y is non-abelian. Then there exists $x \in H$ such that $[W_i, W_i^x] \neq 1$. We may assume that $|W_i / C_{W_i}(W_i^x)| \leq |W_i^x / C_{W_i^x}(W_i)|$. Hence $a(S, W_i) \leq 1$ which contradicts 3.1. ∎

3.3: *Suppose that $(\tau, H) \in \mathcal{F}_2$ and H is solvable. Then $[W, O^2(H)] = 1$.*

Proof: Let $(\tau, H) \in \mathcal{F}_2$ be a counterexample such that $|H|$ is minimal. Then by 1.5, $N_H(W)$ is the unique maximal subgroup of H containing S. Note that by 3.2, $W \not\leq O_2(H)$. Hence, by 2.4 there exists $L \leq H$ satisfying

(*) $L/O_2(L) \cong D_{2r}$, r an odd prime, and

(**) $S \cap L \in \mathrm{Syl}_2(L)$ and $L = \langle W, W^x \rangle$ for $x \in L \setminus N_H(W)$.

We choose the following notation:

$A = W \cap O_2(L)$, $B = W^x \cap O_2(L)$, $X = AB$, $D = A \cap B$, $\overline{X} = X/D$, and $\overline{L} = L/X$.

Note that $O_2(L) = X$. Let j be minimal with the following property:

(***) There exists $y \in G_1$ such that $W_j^y \not\leq O_2(L)$ and $X \leq S^y$.

Note that $y = 1$ and $W_j = W$ satisfy (***). We set $U = H_j^y$, $V_0 = W_j^y$, $V = [V_0, O^2(U)]$, $V_1 = [V, O_2(U)]$, $\overline{V} = V/V_1$, $X_1 = \langle (B \cap O_2(U))^L \rangle$.

Assume that $j = 0$. Then $[X, Z_0^y] = 1$ and thus $[X, O^2(L)] = 1$. On the other hand $[O_2(H), O^2(L)] \leq X$, since $O_2(H) \leq N_H(W) \cap N_H(W^x)$. Hence, $[O_2(H), O^2(L)] = 1$ which contradicts $C_H(O_2(H)) \leq O_2(H)$.

Note that $V_0 = \langle W_{j-1}^{yz} | z \in U \rangle$ since $j > 0$. Hence, there exists $u \in U$ such that $W_{j-1}^{yu} \not\leq O_2(L)$. Note further that $A \leq O_2(U)$ since W is normal in G_1. If $B \leq O_2(U)$, then $X \leq O_2(U) \leq S^{yu}$, which contradicts the minimality of j. Since $BO_2(U) = XO_2(U)$ we have shown:

(1) $B \not\leq O_2(U)$ and $B/B \cap O_2(U) \cong X/X \cap O_2(U)$.

Since $W_{j-1}^y \neq W_j^y$ we get $V \neq 1$. Moreover, $W_{j-1}^y \leq A$ by the minimality of j, and $V_0 = VW_{j-1}^y$. It follows that

(2) $V_0 = V(V_0 \cap A)$, $V \not\leq O_2(L)$ and $|V/V \cap A| = 2$.

Since V operates quadratically on $\overline{X}$ we get from 2.5 that $A = (V \cap A)D$. Note further that B is abelian and $D \leq Z(L)$. Thus, we have

(3) $[X, B \cap O_2(U)] = [A \cap V, B \cap O_2(U)] = [X, X_1] \leq D \cap V_1$.

In particular $|\overline{V}/C_{\overline{V}}(X_1)| \leq |V/V \cap A| = 2$, and 2.6 gives $X_1 \leq O_2(U)$ and $B \cap X_1 = B \cap O_2(U)$. Since by (3) $[B, V \cap X_1] \leq V_1$ we conclude that $\overline{V \cap X_1} \leq C_{\overline{V}}(B)$. This implies that

$$|B/B \cap O_2(U)| = |BX_1/X_1| = |AX_1/X_1| = \frac{1}{2}|VX_1/X_1| \geq \frac{1}{2}|\overline{V}/C_{\overline{V}}(B)|.$$

On the other hand, 2.6 yields $|B/B \cap O_2(U)|^2 \leq |\overline{V}/C_{\overline{V}}(B)|$. Thus, $|\overline{V}/C_{\overline{V}}(B)| \leq 2$, and again 2.6 shows that $B \leq O_2(U)$, which contradicts (1). ∎

3.4: *Suppose that $(\tau, H) \in \mathcal{F}_2$. Then $[W, O^2(H)] = 1$.*

Proof: Let (τ, H) be a counterexample such that $|H|$ is minimal. By 3.2 $W \not\leq O_2(H)$ and by 3.3 H is not solvable. As in 3.3 the minimality of H shows that $N_H(W)$ is the unique maximal subgroup of H containing S. It follows from [Hu,V.25.1 and 25.3] that $O^2(H/O_2(H)) = E_1 \times \cdots \times E_k$, $E_j \cong \mathrm{Sz}(2^m)$, $m > 1$. In particular $N_H(W) = N_H(S \cap O^2(H)O_2(H))$.

Since the centralizer of an involution in $\mathrm{Sz}(2^m)$ is a 2-group we conclude that any two conjugates of W generate a non-solvable group. Hence, by 2.4 there exists $L \leq H$ satisfying

$$(*) \qquad L/O_2(L) \cong \mathrm{Sz}(2^m), \quad m > 1,$$

and

$$(**) \qquad S \cap L \in \mathrm{Syl}_2(L) \quad \text{and} \quad L = \langle W, W^x \rangle \quad \text{for } x \in L \setminus N_H(W).$$

Let X be as in 3.3. The action of $N_L(W)$ on W shows that $L = O^2(L)X$. Hence, L/X is a perfect central extension of $\mathrm{Sz}(2^m)$, and 2.7 implies that $X \leq O_2(L) \leq WX$. This gives $X = O_2(L)$, and by 2.5 $X/Z(L)$ is the direct product of natural $\mathrm{Sz}(2^m)$-modules. Thus, L satisfies the hypothesis of 2.8, but 2.8 contradicts 3.1. ∎

The proof of the Theorem: Define $W(S) := W$. According to 3.4, $W(S)$ is normal in H for every $(\tau, H) \in \mathcal{F}^* \cap \mathcal{C}_M$. Now 1.5 yields the Theorem. ∎

ACKNOWLEDGEMENT: I would like to thank A. Chermak, A. Delgado and G. Glauberman for reading a first draft of this paper and making several helpful suggestions.

References

[Gl₁] G. Glauberman, *Subgroups of finite groups*, Bulletin of the American Mathematical Society **73** (1967), 1–12.

[Gl₂] G. Glauberman, *Factorizations in local subgroups of finite groups*, Regional Conference in Mathematics, Vol. 33 (1977).

[Go] D. Gorenstein, *Finite Groups*, Harper and Row, New York, 1967.

[Gol] D. Goldschmidt, *2-Fusion in finite groups*, Annals of Mathematics **99** (1974), 70–117.

[Hu] B. Huppert, *Endliche Gruppen I*, Springer-Verlag, Berlin, 1967.

[Ma] R. P. Martineau, *On 2-modular representations of the Suzuki groups*, American Journal of Mathematics **94** (1972), 55–72.

[Se] J.-P. Serre, *Trees*, Springer-Verlag, Berlin, 1980.

[St₁] B. Stellmacher, *An analogue to Glauberman's ZJ-Theorem*, Proceedings of the American Mathematical Society **109** (1990), 925–929.

[St₂] B. Stellmacher, *On the 2-local structure of finite groups*, London Mathematical Society Lecture Notes Series **165** (1990), 159–182.

[Su] M. Suzuki, *On a class of doubly transitive groups*, Annals of Mathematics **75** (1962), 105–145.